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Relaxation in the Cauchy problem for Hamilton-Jacobi equations

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1. Introduction. In this note we study a little further the *relaxation* of Hamilton-Jacobi equations developed recently in [4,5]. In [4] we initiated the study of the relaxation of Hamilton-Jacobi equations of eikonal type and in [5] we extended this study to a larger class of Hamilton-Jacobi equations.

Let us recall the relaxation in calculus of variations. In general a non-convex variational problem (P) does not have its minimizer. A natural way to attack such a variational problem is to introduce its relaxed (or convexified) variational problem (RP) which has a minimizer and to regard such a minimizer as a generalized solution of the original problem (P). The main result (or principle) in this direction states that $\min(\mathrm{RP})=\inf(\mathrm{P})$. That is, any accumulation point of a minimizing sequence of (P) is a minimizer of (RP). This fact or principle is called the relaxation of non-convex variational problems. See [3] for a treatment of the relaxation of non-convex variational problems.

Relaxation of Hamilton-Jacobi equations is the principle which says that the pointwise supremum over a suitable collection of Lipschitz continuous subsolutions in the almost everywhere sense of a non-convex Hamilton-Jacobi equation yields a viscosity solution of the equation with convexified Hamiltonian. See [4,5].

Here we are concerned with the Cauchy problem for Hamilton-Jacobi equations and generalize some results obtained in [5].

2. Main result for the Cauchy Problem. We consider the Cauchy Problem

$$u_t(x,t)+H(x,D_xu(x,t))=0 \quad \text{for } (x,t)\in\mathbf{R}^n\times(0,T), \tag{1}$$

$$u|_{t=0}=g, \tag{2}$$

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where H and g are given continuous functions respectively on $\mathbf{R}^{2n}$ and $\mathbf{R}^n$, T is a given positive number or $T = \infty$, $u = u(x,t)$ is the unknown continuous function on $\mathbf{R}^n \times [0,T)$, u_t denotes the t-derivative of u, and $D_x u$ denotes the x-gradient of u.

Let $\widehat{H}$ denote the convex envelope of the function H, that is,

$$\widehat{H}(x,p) = \sup\{l(p) \mid l \text{ affine function, } l(q) \le H(x,q) \text{ for } q \in \mathbf{R}^n\}.$$

We also consider the convexified Hamilton-Jacobi equation

$$u_t(x,t) + \widehat{H}(x, D_x u(x,t)) = 0 \quad \text{for } (x,t) \in \mathbf{R}^n \times (0,T). \tag{3}$$

We use the notation: for $a \in \mathbf{R}^n$ and $r \ge 0$, $B^n(a,r)$ denotes the n-dimensional closed ball of radius r centered at a. For $\Omega \subset \mathbf{R}^m$, $\mathrm{BUC}(\Omega)$ and $\mathrm{UC}\,(\Omega)$ denote the spaces of bounded uniformly continuous functions on Ω and of uniformly continuous functions on Ω, respectively. Furthermore, $\mathrm{Lip}\,(\Omega)$ denotes the space of Lipschitz continuous functions on Ω. Notice that $f \in \mathrm{Lip}\,(\Omega)$ is not assumed to be a bounded function.

Throughout this note we assume:

(4) $H,\ \widehat{H} \in \mathrm{BUC}(\mathbf{R}^n \times B^n(0,R))$ for all $R > 0$.

(5) $\displaystyle \lim_{R\to\infty} \inf\Big\{\frac{H(x,p)}{|p|} \;\Big|\; (x,p) \in \mathbf{R}^n \times (\mathbf{R}^n \setminus B^n(0,R))\Big\} > 0.$

For $R > 0$ we define the function $H_R : \mathbf{R}^{2n} \to \mathbf{R} \cup \{\infty\}$ by

$$H_R(x,p) = \begin{cases} H(x,p) & \text{if } x \in B^n(0,R), \\ \infty & \text{if } x \notin B^n(0,R), \end{cases}$$

and write $\widehat{H}_R$ for $\widehat{G}$, where $G = H_R$.

(6) For each $R > 0$ and $\varepsilon > 0$ there is a constant $\rho \ge R$ such that

$$\widehat{H}_\rho(x,p) \le \widehat{H}(x,p) + \varepsilon \quad \text{for } (x,p) \in \mathbf{R}^n \times B^n(0,R).$$

(7) $g \in \mathrm{UC}\,(\mathbf{R}^n)$.

Proposition 1. (i) *If $u \in \mathrm{USC}(\mathbf{R}^n \times [0,T))$ and $v \in \mathrm{LSC}(\mathbf{R}^n \times [0,T))$ are a viscosity subsolution and a viscosity supersolution of* (3) *respectively. Assume that $u(x,0) \le v(x,0)$ for $x \in \mathbf{R}^n$ and that there is a (concave) modulus ω such that for all $(x,t) \in \mathbf{R}^n \times [0,T)$ and $y \in \mathbf{R}^n$,*

$$\begin{cases} u(x,t) \le u(y,0) + \omega(|x-y| + t), \\ v(x,t) \ge v(y,0) - \omega(|x-y| + t). \end{cases}$$

Then $u \le v$ on $\mathbf{R}^n \times [0,T)$. (ii) *There is a (unique) viscosity solution $u \in \mathrm{UC}\,(\mathbf{R}^n \times [0,\infty))$ of* (3) *which satisfies* (2). *If, in addition, $g \in \mathrm{Lip}\,(\mathbf{R}^n)$, then $u \in \mathrm{Lip}\,(\mathbf{R}^n \times [0,\infty))$.*

We remark that the same proposition as above is valid for (1). We omit giving the proof of the above proposition.

Let $\mathcal{V}_T$ denote the set of functions $v \in \operatorname{Lip}(\mathbf{R}^n \times [0,T))$ such that

$$(8) \qquad v_t(x,t) + H(x, D_x v(x,t)) \le 0 \quad \text{a.e. } (x,t) \in \mathbf{R}^n \times (0,T).$$

The following theorem is the main result in this note.

Theorem 2. *Assume that* (4)–(7) *hold. Let* $u \in \mathrm{UC}(\mathbf{R}^n \times [0,T))$ *be the unique viscosity solution of* (3) *satisfying* (2). *Then, for* $(x,t) \in \mathbf{R}^n \times [0,T)$,

$$(9) \qquad u(x,t) = \sup\{v(x,t) \mid v \in \mathcal{V}_T,\ v|_{t=0} \le g\}.$$

Remark. In general the above formula does not give a subsolution of

$$u_t(x,t) + H(x, D_x u(x,t)) = 0 \quad \text{a.e. } (x,t) \in \mathbf{R}^n \times (0,\infty).$$

For instance, let $n = 2$ and define $H \in C(\mathbf{R}^2)$ and $g \in \mathrm{UC}(\mathbf{R}^2)$ by $H(p,q) = (|p|^{\frac{1}{2}} + |q|^{\frac{1}{2}})^2$ and $g(x,y) = -|x| - |y|$, respectively. Note that $\widehat{H}(p,q) = |p| + |q|$ for $(p,q) \in \mathbf{R}^2$. We set $\rho(x,y,t) = -2t - |x| - |y|$. Then, for instance, by computing $D^{\pm}\rho(x,y,t)$, we infer that ρ is the viscosity solution of

$$\begin{cases} u_t(x,y,t) + |u_x(x,y,t)| + |u_y(x,y,t)| = 0 & \text{in } \mathbf{R}^2 \times (0,\infty), \\ u(x,y,0) = g(x,y) \quad \text{for } (x,y) \in \mathbf{R}^2. \end{cases}$$

On the other hand, since at any point $(x,y,t) \in \mathbf{R}^2 \times (0,\infty)$, where x, $y \ne 0$, we have

$$H(\rho_x(x,y,t), \rho_y(x,y,t)) = 4, \qquad \rho_t(x,y,t) = -2,$$

ρ is not a subsolution of

$$u_t(x,y,t) + (|u_x(x,y,t)|^{\frac{1}{2}} + |u_y(x,y,t)|^{\frac{1}{2}})^2 = 0 \quad \text{a.e. } (x,y,t) \in \mathbf{R}^n \times (0,\infty).$$

Theorem 2 is an easy consequence of the following theorem.

Theorem 3. *Assume that* (4)–(6) *hold. Let* $u \in \mathrm{UC}(\mathbf{R}^n \times [0,T))$ *be a viscosity subsolution of* (3). *Then, for all* $(x,t) \in \mathbf{R}^n \times [0,T)$,

$$(10) \qquad u(x,t) = \sup\{v(x,t) \mid v \in \mathcal{V}_T,\ v \le u \ \text{ in } \mathbf{R}^n \times [0,T)\}.$$

Conceding Theorem 3 for the moment, we finish the proof of Theorem 2 as follows.

Proof of Theorem 2. We write $w(x,t)$ for the right hand side of (9). By Theorem 3 we find that $u \le w$ on $\mathbf{R}^n \times [0,T)$. Let $v \in \mathcal{V}_T$ satisfy $v(\cdot,0) \le g$ on $\mathbf{R}^n$. Then, since $\widehat{H} \le H$, we have

$$v_t(x,t) + \widehat{H}(x, D_x v(x,t)) \le 0 \quad \text{a.e. } (x,t) \in \mathbf{R}^n \times (0,T).$$

Since $\widehat{H}(x,\cdot)$ is convex, v is a viscosity subsolution of (3). By (i) of Proposition 1, we have $v \le u$ on $\mathbf{R}^n \times (0,T)$, from which we get $w \le u$ on $\mathbf{R}^n \times (0,T)$. Thus we have $u = w$ on $\mathbf{R}^n \times (0,T)$. $\square$

For our proof of Theorem 3, we need several lemmas. For a proof of the next three lemmas, we refer to [5].

Lemma 4. *Let K be a non-empty convex subset of $\mathbf{R}^m$ and set*

$$L(\xi) = \sup\{\xi \cdot p \mid p \in K\} \in \mathbf{R} \cup \{\infty\} \quad \textit{for all } \xi \in \mathbf{R}^m.$$

Let U be an open subset of $\mathbf{R}^m$ and let $v \in C(\overline{U})$ satisfy

$$D^+v(x) \subset K \quad \textit{for all } x \in U.$$

Let $x, y \in \overline{U}$, and assume that the open line segment $l_0(x,y) := \{tx + (1-t)y \mid t \in (0,1)\} \subset U$. Then

$$u(x) \le u(y) + L(x-y).$$

In the above lemma and in what follows, for $v \in C(U)$ and $x \in U$, $D^+v(x)$ denotes the superdifferential of v at x.

Lemma 5. *Let Ω be an open subset of $\mathbf{R}^m$ and $f_1, ..., f_N \in \mathrm{Lip}(\Omega)$, with $N \in \mathbf{N}$. Set*

$$f(x) = \max\{f_1(x), ..., f_N(x)\} \quad \textit{for } x \in \Omega.$$

Then $f \in \mathrm{Lip}(\Omega)$ and f, $f_1, ..., f_N$ are almost everywhere differentiable. Moreover for almost every $x \in \Omega$,

$$Df(x) \in \{Df_1(x), ..., Df_N(x)\},$$

where $Df(x)$ denotes the gradient of f at x.

Lemma 6. *Let Z be a non-empty closed subset of $\mathbf{R}^m$. Define $L : \mathbf{R}^m \to \mathbf{R} \cup \{\infty\}$ by*

$$L(\xi) = \sup\{\xi \cdot p \mid p \in Z\}.$$

Let $\bar{\xi} \in \mathbf{R}^m$ be a point where L is differentiable. Then

$$DL(\bar{\xi}) \in Z \cap \partial(\overline{\mathrm{co}}\, Z)$$

We introduce the notation: for $(x,r) \in \mathbf{R}^n \times \mathbf{R}$ let

$$Z(x,r) := \{(p,q) \in \mathbf{R}^{n+1} \mid q + H(x,p) \leq r\}$$

and $K(x,r) := \overline{\mathrm{co}}\, Z(x,r)$, the closed convex hull of $Z(x,r)$. We note that

$$K(x,r) = \{(p,q) \in \mathbf{R}^{n+1} \mid q + \widehat{H}(x,p) \leq r\}.$$

For $\delta > 0$, let $\Delta(\delta) := \{(x,y) \in \mathbf{R}^{2n} \mid |x-y| \leq \delta\}$.

Lemma 7. *Assume that* (4) *holds. For any $R > 0$ and $\varepsilon > 0$ there exists a constant $\delta > 0$ such that for any $(x,y) \in \Delta(\delta)$ and $r \in \mathbf{R}$,*

$$Z_R(x,r) + B^{n+1}(0,\delta) \subset Z_{R+1}(y, r+\varepsilon),$$

where, for $R > 0$, $Z_R(x,r) = Z(x,r) \cap B^{n+1}(0,R)$.

Proof. Fix $\varepsilon > 0$ and $R > 0$. Let ω denote the modulus of continuity of H on $\mathbf{R}^n \times B^n(0, R+1)$.

Fix a constant $\delta \in (0,\, 1)$ so that $\delta + \omega(2\delta) \leq \varepsilon$. Fix $(\xi,\eta) \in B^{n+1}(0,\delta)$, $(x,y) \in \Delta(\delta)$, $(p,q) \in Z_R(x,0)$, and $r \in \mathbf{R}$.

Noting that $(p,q) + (\xi,\eta) \in B^{n+1}(0, R+1)$, we observe that

$$q + \eta + H(y, p+\xi) \leq q + H(x,p) + \eta + \omega(|x-y| + |\xi|) \leq r + \delta + \omega(2\delta) \leq r + \varepsilon.$$

Thus we have

$$(p+\xi, q+\eta) \in Z_{R+1}(y, r+\varepsilon),$$

which concludes the proof. $\square$

Lemma 8. *Assume that* (4)–(6) *hold. For any $R > 0$ and $\varepsilon > 0$ there exists a constant $M \geq R$ such that for any $x \in \mathbf{R}^n$,*

$$K_R(x,0) \subset \mathrm{co}\, Z_M(x,\varepsilon),$$

where $K_R(x,r) = K(x,r) \cap B^{n+1}(0,R)$.

Proof. For $R > 0$ and $\varepsilon > 0$ let $\rho \equiv \rho(R,\varepsilon) \geq R$ be the constant from (6). That is, $\rho = \rho(R,\varepsilon)$ is a constant for which

$$\widehat{H}_\rho(x,p) \leq \widehat{H}(x,p) + \varepsilon \quad \text{for } (x,p) \in \mathbf{R}^n \times B^n(0,R).$$

In view of (4), for $R > 0$ let $M_R \geq 0$ be the constant defined by

$$M_R = \sup\{|H(x,p)| \mid (x,p) \in \mathbf{R}^n \times B^n(0,R)\}.$$

Fix $R > 0$, $\varepsilon > 0$, $x \in \mathbf{R}^n$, and $(p,q) \in K_R(x,0)$. We have

$$\widehat{H}(x,p) + q \leq 0,$$

and hence

$$\widehat{H}_\rho(x,p) + q \leq \varepsilon.$$

Choose sequences $\{\lambda_i\}_{i=1}^m \subset (0,1]$ and $\{p_i\}_{i=1}^m \subset B^n(0,\rho)$, with $m \in \mathbf{N}$, so that

$$\sum_{i=1}^m \lambda_i p_i = p, \qquad \sum_{i=1}^m \lambda_i = 1,$$
$$\sum_{i=1}^m \lambda_i H(x,p_i) + q \leq 2\varepsilon.$$

(See the proof of Lemma 10 below.) Setting

$$h = q + \sum_{i=1}^m \lambda_i H(x,p_i), \qquad q_i = h - H(x,p_i) \quad \text{for } i = 1,2,...,m,$$

we observe that

$$h \leq 2\varepsilon, \qquad h \geq -|q| - M_\rho \geq -R - M_\rho,$$
$$|q_i| \leq |h| + M_\rho \leq 2\varepsilon + R + 2M_\rho \quad \text{for } i = 1,2,...,m,$$

and that

$$(p_i,q_i) \in Z(x,h) \subset Z(x,2\varepsilon) \quad \text{for } i = 1,2,...,m,$$
$$\sum_{i=1}^m \lambda_i q_i = h - \sum_{i=1}^m \lambda_i H(x,p_i) = q,$$
$$\sum_{i=1}^m \lambda_i (p_i,q_i) = (p,q).$$

These together show that $(p,q) \in \operatorname{co} Z_M(x,2\varepsilon)$, with $M = (\rho^2 + (2\varepsilon + R + 2M_\rho)^2)^{1/2}$. □

Proof of Theorem 3. We write $Q = \mathbf{R}^n \times (0,T)$ and $Q_\delta = \mathbf{R}^n \times (-\delta, T+\delta)$ for $\delta > 0$.

Firstly, without loss of generality we may assume that u is defined and Lipschitz continuous on Q_δ for some constant $\delta > 0$ and that

$$u_t(x,t) + \widehat{H}(x, D_x u(x,t)) \leq 0 \quad \text{in } Q_\delta \tag{11}$$

in the viscosity sense. Indeed, we have

$$u(x,t) = \sup\{v(x,t) \mid v \in \mathrm{Lip}(Q_\delta) \text{ for some } \delta > 0, \\ v \text{ is a viscosity solution of (11)}, \ v \le u \text{ on } Q\}. \tag{12}$$

To see this, assuming $T < \infty$, we solve the Cauchy problem

$$w_t(x,t) + \widehat{H}(x, D_x w(x,t)) \le 0 \quad \text{in } \mathbf{R}^n \times (T, T+1)$$

with the initial condition

$$w(x,T) = \lim_{t \nearrow T} u(x,t) \quad \text{for } x \in \mathbf{R}^n. \tag{13}$$

In view of (4) and (5), there is a constant $C > 0$ such that $\widehat{H}(x,p) \ge -C$ for all $(x,p) \in \mathbf{R}^{2n}$, which shows that u is a viscosity solution of $u_t \le C$ in $\mathbf{R}^n \times (0,T)$. This monotonicity of the function $u(x,t)$ in t and the uniform continuity of u guarantee that the limit on the right hand side of (13) defines a uniform continuous function on $\mathbf{R}^n$.

By (ii) of Proposition 1, there is a unique viscosity solution $w \in \mathrm{UC}(\mathbf{R}^n \times [T, T+1))$ for which (13) holds. We extend the domain of definition of w to $\mathbf{R}^n \times (0, T+1)$ by setting

$$w(x,t) = u(x,t) \quad \text{for } (x,t) \in \mathbf{R}^n \times (0,T).$$

It is easy to see that $w \in \mathrm{UC}(\mathbf{R}^n \times (0, T+1))$ that w is a viscosity subsolution of

$$w_t(x,t) + \widehat{H}(x, D_x w(x,t)) = 0 \quad \text{in } \mathbf{R}^n \times (0, T+1).$$

Now, if $T = \infty$, we define $w \in \mathrm{UC}\,(\mathbf{R}^n \times [0,\infty))$ by setting $w = u$.

Fix any $\varepsilon > 0$. Since $w \in \mathrm{UC}(\mathbf{R}^n \times (0, T+1))$, there is a constant $\delta \in (0, 1/2)$ such that

$$u(x,t) - 2\varepsilon \le w(x, t-\delta) - \varepsilon \le u(x,t) \quad \text{for } (x,t) \in \mathbf{R}^n \times (0,T). \tag{14}$$

It is clear that the function $z(x,t) := w(x, t-\delta) - 2\varepsilon$ is defined and uniformly continuous on Q_δ and is a viscosity solution of (11).

Now, we take the sup-convolution of z in the t-variable. That is, for $\gamma > 0$, we consider the function

$$z^\gamma(x,t) = \sup\{z(x,s) - \frac{1}{2\gamma}(t-s)^2 \mid s \in (-\delta, T+\delta)\} \quad \text{for } (x,t) \in \mathbf{R}^{n+1}.$$

If $\gamma > 0$ is small enough, then z^γ is a viscosity solution of (11) in $Q_{\delta/2}$ and

$$z(x,t) \le z^\gamma(x,t) \le z(x,t) + \varepsilon \quad \text{for } (x,t) \in Q_\delta. \tag{15}$$

Note also that, for each $\gamma > 0$, the collection of functions $z^\gamma(x, \cdot)$, with $x \in \mathbf{R}^n$, is equi-Lipschitz continuous on $(-\delta/2, T + \delta/2)$. By virtue of (5), we may choose constants $c_0 > 0$ and $C_1 > 0$ such that

$$\widehat{H}(x,p) \geq c_0|p| - C_1 \quad \text{for } (x,p) \in \mathbf{R}^{2n}.$$

Since z^γ is a viscosity solution of

$$c_0|D_x z^\gamma(x,t)| \leq C_1 + L_\gamma \quad \text{in } Q_{\delta/2},$$

where $L_\gamma > 0$ is a uniform Lipschitz bound of the functions $z^\gamma(x, \cdot)$ on $(-\delta/2, T+\delta/2)$, we see that the functions $z^\gamma(\cdot, t)$ are Lipschitz continuous on $\mathbf{R}^n$, with a Lipschitz bound independent of $t \in (-\delta/2, T + \delta/2)$.

Now, using (14) and (15) and writing $U(x,t)$ for the right hand side of (12), we see that for sufficiently small $\gamma > 0$ and for all $(x,t) \in Q$,

$$u(x,t) \geq z(x,t) + \varepsilon \geq z^\gamma(x,t),$$

and hence,

$$U(x,t) \geq z^\gamma(x,t) \geq z(x,t) \geq u(x,t) - 3\varepsilon,$$

which proves (12).

Henceforth we assume that, for some constant $\delta > 0$, u is a member of $\mathrm{Lip}(Q_\delta)$ and satisfies (11) in the viscosity sense.

Let $R > 0$ be a Lipschitz bound of the function u. Fix any $\varepsilon \in (0,\ 1)$. Due to Lemma 8, there is a constant $\rho \geq R$ such that for all $x \in \mathbf{R}^n$,

$$K_R(x,0) \subset \operatorname{co} Z_\rho(x,\varepsilon).$$

In view of Lemma 7, there is a constant $\gamma \in (0,1)$ such that for any $(x,y) \in \Delta(\gamma)$,

$$Z_\rho(x,\varepsilon) + B^{n+1}(0,\gamma) \subset Z_{\rho+1}(y, 2\varepsilon).$$
$$Z_{\rho+1}(y,2\varepsilon) \subset Z_{\rho+2}(x, 3\varepsilon).$$

Consequently, for $(x,y) \in \Delta(\gamma)$, we have

$$K_R(x,0) + B^{n+1}(0,\gamma) \subset \operatorname{co} Z_{\rho+1}(y, 2\varepsilon), \tag{16}$$
$$Z_{\rho+1}(y,2\varepsilon) \subset Z_{\rho+2}(x,3\varepsilon). \tag{17}$$

We may assume that $\gamma < \delta$. Let $\mu \in (0,\gamma)$ be a constant to be fixed later. We choose a set $Y_\mu \subset Q_\delta$ so that

$$\#(Y_\mu \cap B^{n+1}(0,r)) < \infty \quad \text{for all } r > 0, \tag{18}$$
$$\bigcup_{(y,s)\in Y_\mu} B^{n+1}((y,s),\mu) \supset Q_\delta. \tag{19}$$

We set

$$L(\xi,\eta;y)=\sup\{\xi\cdot p+\eta q \mid (p,q)\in Z_{\rho+1}(y,2\varepsilon)\} \quad \text{for } \xi,y\in\mathbf{R}^n,\ \eta\in\mathbf{R}$$

and

$$v(x,t;y,s)=u(y,s)+L(x-y,t-s;y) \quad \text{for } (x,t)\in\mathbf{R}^{n+1},\ (y,s)\in Q_\delta.$$

By Lemma 6, we get for $(x,y)\in\Delta(\gamma)$,

$$D_{\xi,\eta}L(\xi,\eta;y)\in Z_{\rho+1}(y,2\varepsilon)\subset Z_{\rho+2}(x,3\varepsilon) \quad \text{a.e. } (\xi,\eta)\in\mathbf{R}^{n+1}. \tag{20}$$

Noting that

$$D^+u(x,t)\subset K_R(x,0) \quad \text{for } (x,t)\in Q_\delta,$$

and setting $\tilde{u}(x,t):=u(x,t)+\gamma|(x,t)-(y,s)|$ for $(x,t),(y,s)\in Q_\delta$, we find that for $(x,t),(y,s)\in Q_\delta$, if $0<|x-y|\le\gamma$, then

$$D^+\tilde{u}(x,t)\subset D^+u(x,t)+B^{n+1}(0,\gamma)\subset \operatorname{co} Z_{\rho+1}(y,2\varepsilon).$$

Hence, by Lemma 4, we get

$$u(x,t)+\gamma|(x,t)-(y,s)|\le v(x,t;y,s) \quad \text{for } (x,t),(y,s)\in Q_\delta, \text{ with } |x-y|\le\delta. \tag{21}$$

Set $\beta=\gamma/5$ and define the function $w:Q_{2\beta}\to\mathbf{R}$ by

$$w(x,t)=\min\{v(x,t;y,s) \mid (y,s)\in Y_\mu\cap B^{n+1}((x,t),3\beta)\}.$$

Now, we show that if μ is sufficiently small, then for $(\bar{x},\bar{t})\in Q_\beta$ and $(x,t)\in B^{n+1}((\bar{x},\bar{t}),\beta)$

$$w(x,t)=\min\{v(x,t;y,s) \mid (y,s)\in Y_\mu\cap B^{n+1}((\bar{x},\bar{t}),2\beta)\}. \tag{22}$$

To do this, fix $(\bar{x},\bar{t})\in Q_\beta$ and $(x,t)\in Y_\mu\cap B^{n+1}((\bar{x},\bar{t}),2\beta)$. Noting that $Y_\mu\cap B^{n+1}((x,t),\mu)\neq\emptyset$ and $B^{n+1}((x,t),\mu)\subset B^{n+1}((x,t),5\beta)$ and choosing a point $(y,s)\in Y_\mu\cap B^{n+1}((x,t),\mu)$, we see that

$$\begin{aligned} w(x,t) &\le v(x,t;y,s)\le u(y,s)+(\rho+1)|(x,t)-(y,s)| \\ &\le u(x,t)+(R+\rho+1)|(x,t)-(y,s)|. \end{aligned}$$

Here we have used the fact that the functions $L(\xi,\eta;y)$ of (ξ,η) are Lipschitz continuous functions with $\rho+1$ as a Lipschitz bound. Fix now $\mu\in(0,\gamma)$ by setting

$$\mu=\frac{1}{2}\min\{\gamma,\frac{\gamma\beta}{R+\rho+1}\}$$

and observe that

$$w(x,t) < u(x,t) + \gamma\beta. \tag{23}$$

Fix $(y,s) \in Q_\delta \setminus B^{n+1}((\bar{x},\bar{t}),2\beta)$ and note that $|(y,s)-(x,t)| \geq \beta$. Using (21), we have

$$v(x,t;y,s) \geq u(x,t) + \gamma\beta.$$

From this and (23), we conclude that (22) holds.

Next, we observe from (22) that the function w is Lipschitz continuous on $B^{n+1}((\bar{x},\bar{t}),\beta)$ for all $(\bar{x},\bar{t}) \in Q_\beta$, with $\rho+1$ as a Lipschitz bound, which guarantees that $w \in \mathrm{Lip}\,(Q_\beta)$. Applying Lemma 5 and using (20), we observe that w is almost everywhere differentiable on Q_β and, at any point $(x,t) \in Q_\beta$ where w is differentiable,

$$Dw(x,t) \in \bigcup\{D_{x,t}v(x,t;y,s) \mid (y,s) \in Y_\mu \cap B^{n+1}((\bar{x},\bar{t}),2\beta)\} \subset Z_{\rho+2}(x,3\varepsilon),$$

which yields readily

$$w_t(x,t) + H(x, D_x w(x,t)) \leq 3\varepsilon \quad \text{a.e. } (x,t) \in Q_\beta.$$

Setting

$$z(x,t) = w(x,t) - \gamma\beta - 3\varepsilon t \quad \text{for } (x,t) \in Q_\beta,$$

we have

$$z_t(x,t) + H(x, D_x z(x,t)) \leq 0 \quad \text{a.e. } (x,t) \in Q_\beta.$$

By (23), we have $z(x,t) \leq u(x,t) - 3\varepsilon t$ for $(x,t) \in Q_\beta$ and, by (21), we have $z(x,t) \geq u(x,t) - \gamma\beta - 3\varepsilon t$ for $(x,t) \in Q_\beta$. In the above two inequalities, we may take $\gamma > 0$ as small as we wish. Thus we get

$$u(x,t) = \sup\{z(x,t) \mid z \in \mathcal{V}_T,\ z \leq u \text{ on } Q\} \quad \text{for } (x,t) \in Q,$$

which completes the proof. □

3. Examples. In this section we consider some examples of Hamiltonians H and examine if H satisfies conditions (4)–(6) or not.

Let $H \in C(\mathbf{R}^{2n})$ be a function of the form

$$H(x,p) = G(x,p)^m + f(x),$$

where $G \in C(\mathbf{R}^{2n})$ satisfies

$$G \in \mathrm{BUC}(\mathbf{R}^n \times B^n(0,R)) \quad \text{for } R > 0, \tag{24}$$

$$G(x,\lambda p) = \lambda G(x,p) \quad \text{for } \lambda \geq 0, (x,p) \in \mathbf{R}^{2n}, \tag{25}$$

$$\delta_G := \inf_{\mathbf{R}^n \times \partial B^n(0,1)} G > 0. \tag{26}$$

m is a constant satisfying $m \geq 1$, and $f \in \mathrm{BUC}(\mathbf{R}^n)$.

Proposition 9. *The function H given above satisfies* (4)–(6).

We need the following Lemma.

Lemma 10. *For all $(x,p) \in \mathbf{R}^{2n}$, we have*

$$\widehat{G}(x,p) = \min\{r \in \mathbf{R} \mid p = \sum_{i=1}^{k} \lambda_i p_i,\ \lambda_i > 0,\ \sum_{i=1}^{k} \lambda_i = 1,\ G(x,p_i) = r\}. \tag{27}$$

Proof. We fix $x \in \mathbf{R}^n$ and write $G(p)$ for $G(x,p)$ for notational simplicity. By using the separation theorem and Carathéodory's theorem in convex analysis, we see easily that

$$\widehat{G}(p) = \inf\{\sum_{i=1}^{n+1} \lambda_i G(p_i) \mid \lambda_i \geq 0,\ \sum_{i=1}^{n+1} \lambda_i = 1,\ \sum_{i=1}^{n+1} \lambda_i p_i = p\} \quad \text{for } p \in \mathbf{R}^n. \tag{28}$$

It is clear from the above representation formula that

$$\widehat{G}(\lambda p) = \lambda \widehat{G}(p) \quad \text{for } (\lambda, p) \in [0,\infty) \times \mathbf{R}^n,$$
$$G(p) \geq \widehat{G}(p) \geq \delta_G |p| \quad \text{for } p \in \mathbf{R}^n.$$

Fix $p \in \mathbf{R}^n$. If $p = 0$, then it is clear that (27) holds. We may thus assume that $p \neq 0$. For any $r > \widehat{G}(p)$, by the above formula, there are $\{\lambda_i\}_{i=1}^{n+1} \subset [0,1]$ and $\{p_i\}_{i=1}^{n+1} \subset \mathbf{R}^n$ such that

$$r > \sum_{i=1}^{n+1} \lambda_i G(p_i), \quad \sum_{i=1}^{n+1} \lambda_i = 1, \quad \sum_{i=1}^{n+1} \lambda_i p_i = p.$$

Set

$$s = \sum_{i=1}^{n+1} \lambda_i G(p_i), \qquad \mu_i = s^{-1} G(p_i).$$

Notice that $s \geq \widehat{G}(p) > 0$ by (28). By rearranging the order in i if necessary, we may assume that

$$\lambda_i \mu_i > 0 \quad \text{for } i \leq k, \qquad \lambda_i \mu_i = 0 \quad \text{for } i > k$$

for some $k \in \{1, ..., n+1\}$. Note that if $i > k$ and $\lambda_i > 0$, then $p_i = 0$. We now have

$$\sum_{i=1}^{k} \lambda_i \mu_i = s^{-1} \sum_{i=1}^{n+1} \lambda_i G(p_i) = 1,$$
$$\sum_{i=1}^{k} \lambda_i \mu_i (\mu_i^{-1} p_i) = \sum_{i=1}^{k} \lambda_i p_i = \sum_{i=1}^{n+1} \lambda_i p_i = p,$$
$$G(\mu_i^{-1} p_i) = s G(p_i)^{-1} G(p_i) = s \quad \text{for } i = 1, ..., k.$$

Hence we get

$$\widehat{G}(p) \geq \inf\{s \in \mathbf{R} \mid \lambda_i > 0,\ G(p_i) = s,\ \sum_{i=1}^{k} \lambda_i p_i = p,\ k \leq n+1\}.$$

Since the set $\{q \in \mathbf{R}^n \mid G(q) \leq \widehat{G}(p) + 1\}$ is a compact set, it is not hard to see that the infimum on the right hand side of the above inequality is actually attained. That is, we have

$$\widehat{G}(p) \geq \min\{s \in \mathbf{R} \mid \lambda_i > 0,\ G(p_i) = s,\ \sum_{i=1}^{k} \lambda_i p_i = p,\ k \leq n+1\}.$$

The opposite inequality is obvious. The proof is now complete. □

Proof of Proposition 9. First we observe that

$$\widehat{H}(x,p) = \widehat{G}(x,p)^m + f(x) \quad \text{for } (x,p) \in \mathbf{R}^{2n}. \tag{29}$$

Indeed, since the function:

$$p \mapsto \widehat{G}(x,p)^m + f(x)$$

is convex on $\mathbf{R}^n$ for every $x \in \mathbf{R}^n$ and

$$\widehat{G}(x,p)^m + f(x) \leq H(x,p) \quad \text{for } (x,p) \in \mathbf{R}^{2n},$$

we see that

$$\widehat{G}(x,p)^m + f(x) \leq \widehat{H}(x,p) \quad \text{for } (x,p) \in \mathbf{R}^{2n}.$$

On the other hand, by Lemma 10, for $(x,p) \in \mathbf{R}^{2n}$ we have

$$\begin{aligned}\widehat{G}(x,p)^m &= \min\{r^m \in \mathbf{R} \mid k \leq n+1,\ \lambda_i > 0,\ G(x,p_i) = r,\ \sum_{i=1}^{k} \lambda_i = 1,\ \sum_{i=1}^{k} \lambda_i p_i = p\} \\ &\geq \inf\{\sum_{i=1}^{k} \lambda_i G(x,p_i)^m \mid k \in \mathbf{N},\ \lambda_i > 0,\ \sum_{i=1}^{k} \lambda_i = 1,\ \sum_{i=1}^{k} \lambda_i p_i = p\}.\end{aligned}$$

Hence, by the formula

$$\widehat{H}(x,p) = \inf\{\sum_{i=1}^{k} \lambda_i H(x,p_i) \mid k \in \mathbf{N},\ \lambda_i > 0,\ \sum_{i=1}^{k} \lambda_i = 1,\ \sum_{i=1}^{k} \lambda_i p_i = p\},$$

we have

$$\widehat{G}(x,p)^m + f(x) \geq \widehat{H}(x,p).$$

Thus we have shown (29).

To show that H satisfies (4), we just need to prove that

$$\widehat{G} \in \mathrm{BUC}(\mathbf{R}^n \times B^n(0,R)) \quad \text{for } R > 0.$$

Fix $R > 0$, set

$$\rho_1 = \sup_{\mathbf{R}^n \times B^n(0,R)} G,$$

and, in view of (26), choose $\rho_2 > 0$ so that

$$\inf_{\mathbf{R}^n \times (\mathbf{R}^n \setminus B^n(0,\rho_2))} G > \rho_1.$$

Then, by Lemma 10, we have

$$\begin{aligned}
\widehat{G}(x,p) &= \min\{\sum_{i=1}^k \lambda_i G(x,p_i) \mid \lambda_i \geq 0,\ \sum_{i=1}^k \lambda_i = 1,\ G(x,p_i) \leq \rho_1,\ \sum_{i=1}^k \lambda_i p_i = p\} \\
&= \min\{\sum_{i=1}^k \lambda_i G(x,p_i) \mid \lambda_i \geq 0,\ \sum_{i=1}^k \lambda_i = 1,\ p_i \in B^n(0,\rho_2),\ \sum_{i=1}^k \lambda_i p_i = p\} \\
&\qquad \text{for } (x,p) \in \mathbf{R}^n \times B^n(0,R).
\end{aligned}$$

This shows that the collection of functions:

$$x \mapsto \widehat{G}(x,p),$$

with $p \in B^n(0,R)$, is equi-continuous on $\mathbf{R}^n$. On the other hand,

$$\{\widehat{G}(x,\cdot) \mid x \in \mathbf{R}^n\}$$

is a uniformly bounded collection of convex functions on $B^n(0,R)$. Consequently, this collection is equi-Lipschitz continuous on $B^n(0,R)$. Thus we see that $\widehat{G} \in \mathrm{BUC}(\mathbf{R}^n \times B^n(0,R))$ for all $R > 0$.

By assumptions (25) and (26), H clearly satisfies (5).

To show (6), fix $R > 0$ and choose $\rho_2 > 0$ as above. Then, by Lemma 10, we get

$$\begin{aligned}
\widehat{G}(x,p)^m = \min\{\sum_{i=1}^k \lambda_i G(x,p_i)^m \mid k \in \mathbf{N},\ \lambda_i \geq 0,\ G(x,p_i) = \widehat{G}(x,p),& \\
\sum_{i=1}^k \lambda_i = 1,\ \sum_{i=1}^k \lambda_i p_i = p\}& \\
= \min\{\sum_{i=1}^k \lambda_i G(x,p_i)^m \mid k \in \mathbf{N},\ \lambda_i \geq 0,\ p_i \in B^n(0,\rho_2),& \\
\sum_{i=1}^k \lambda_i = 1,\ \sum_{i=1}^k \lambda_i p_i = p\}.&
\end{aligned}$$

Hence we have

$$\widehat{H}(x,p) = \widehat{H}_{\rho_2}(x,p) \quad \text{for } (x,p) \in \mathbf{R}^n \times B^n(0,R).$$

Thus H satisfies (4)–(6). □

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